

# Stable Interpolation for the Yield Curve

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(v1.3 released July 2013)

**ABSTRACT** *We study several interpolation schemes that are adequate for the yield curve construction, and pay attention to their stability under sequential and parallel perturbations. It is found that Hagan and West monotone convex interpolation, tension splines and some monotone Hermite spline don't always create stable yield curves. A specific monotone Hermite spline interpolation is however stable.*

**KEY WORDS:** monotone interpolation, cubic spline, yield curve, tension spline, finance

## 1. Introduction

### 1.1 Yield Curve Construction

Let  $P(t)$  denote the price of a zero coupon bond delivering for certain \$1 at maturity  $t$ . The continuously compounded yield is given by:

$$P(t) = \exp(-y(t)t) \quad (1)$$

The logarithm of the discount factor is:

$$z(t) = -y(t)t \quad (2)$$

The instantaneous forward rate is:

$$f(t) = \frac{\partial}{\partial t}(y(t)t) \quad (3)$$

A complete description of yield curve construction is given in Andersen and Piterberg (2010). Let  $V_i$  be the price of  $N$  securities. Typically, for a Libor curve, those securities are Libor deposits for the first few months, Eurodollar futures for up to 3 or 4 years and par swaps for the rest of the curve. We assume that the securities can be written as a linear combination of discount bond prices:

$$V_i = \sum_{j=1}^M c_{i,j} P(t_j) \quad , \quad i = 1, \dots, N \quad (4)$$

with  $t_1, \dots, t_M$  a finite set of dates, in practice corresponding to the cash flow dates of the  $N$  benchmark securities.

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Productive conversations with colleague Gary Goldberg is gratefully acknowledged. Discussions and exchange of ideas with Gary Kennedy were particularly important.

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We will use a parameterization of the yield curve with  $N$  parameters, using the  $N$  equations to solve the parameters, and limit our study to spline-like interpolators, the most common in practice (Andersen and Piterbarg, 2010).

### 1.2 Yield Curve Risk

Let  $\Pi$  be a portfolio of securities. The yield curve risk is directly related to the estimation of the the delta  $\frac{\partial \Pi}{\partial V_i}$ .

The simplest way to estimate the delta is the Par-Point approach where  $V_i$  is bumped (often by bumping the corresponding yield by 1 basis point), the yield curve is then rebuilt, and the portfolio repriced. A variation of the Par-Point approach uses cumulative shifts:  $\Pi(V_1 + \Delta V_1, \dots, V_i + \Delta V_i, V_{i+1}, \dots, V_N) - \Pi(V_1 + \Delta V_1, \dots, V_{i-1} + \Delta V_{i-1}, V_i, V_{i+1}, \dots, V_N)$  instead of a single Par-Point shift. This has the advantage of preserving the parallel delta: the sum of cumulative deltas is by construction the parallel delta, that is the delta following a parallel shift of the benchmark securities. The cumulative shift is also somewhat more realistic.

An alternative is the forward rate approach where the forward curve is directly perturbed, and the curve is therefore not rebuilt. The forward curve shift used in practice are often either piecewise triangular or piecewise flat. The delta corresponding to a forward rate shift often does not translate into a hedge quantity, unlike the Par-Point approach. But the hedge can be recovered via the Jacobian method (Andersen and Piterbarg, 2010). Using this approach, the interpolator has little role in the risk values, as the curve is not rebuilt.

It is common for banks to use both approaches at the same time. We focus our analysis on the Par-Point approach for estimating the delta, not least because of the stress it puts on interpolators.

### 1.3 Good Interpolation

Hagan and West have reviewed a wide range of interpolation schemes for the yield curve construction (Hagan and West, 2006). They judge those according to the following criteria:

- (a) Is the curve arbitrage free? The forwards must be positive.
- (b) How good do the forward rates look? The forwards must be as continuous as possible.
- (c) How local is the interpolation method? if an input is changed, the yield curve should change only nearby, with minor spill-over elsewhere.
- (d) Are the forwards stable? This is quantified by looking at the maximum basis point change in the forward curve given some basis point change in one of the inputs.
- (e) How local are the hedges? Most of the delta risk should be assigned to hedging instruments with maturities close to the given tenor.

On top of those criteria, we add another stability criteria:

- (f) Is the sequential delta close enough to the parallel delta? When the curve inputs are shifted independently quote by quote by some basis point change, is the total price change of a simple linear instrument the same as when the input quotes are shifted together in parallel? This is important for the consistency of risk measures.

The standard linear interpolation on the log of discount factors (known as raw interpolation) fulfill all the criteria except the continuity of the forwards (b). The standard cubic spline fails criteria (a) and (e). With Hyman filtering it still fails (e), as does the more refined monotonic spline interpolation of (Wolberg and Alf, 1999). A Bessel spline on the log of discount factors with Hyman filtering (Hyman, 1983) is found to be a potential candidate in (Ametrano and Bianchetti, 2009). Finally, Andersen proposes hyperbolic tension splines to smoothly manipulate locality and shape preservation (Andersen, 2005).

## 2. Hagan and West monotone convex interpolation issues

### 2.1 The Forward monotone convex spline

Let's recall Hagan and West monotone convex spline construction (Hagan and West, 2006).

Firstly, suitable discrete forward rates are computed according to the following procedure: Let  $f_i^d$  be the input forward rate at node  $i$ , the rate at point  $t_i$  is defined for  $i = 1, 2, \dots, n-1$  by:

$$f_i = \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} f_{i+1}^d + \frac{t_{i+1} - t_i}{t_{i+1} - t_i} f_i^d \quad (5)$$

$$f_0 = f_1^d - \frac{1}{2}(f_1 - f_1^d) \quad (6)$$

$$f_n = f_n^d - \frac{1}{2}(f_{n-1} - f_n^d) \quad (7)$$

Secondly, for  $i = 1, 2, \dots, n$ , let  $g(t) = f(t) - f_i^d$  and  $x(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}}$ . The spline is defined for  $x \in [0, 1]$  by:

$$g(x) = g_{i-1}(1 - 4x + 3x^2) + g_i(-2x + 3x^2)$$

This definition makes  $\frac{1}{t_i-t_{i-1}} \int_{t_{i-1}}^{t_i} f(t) dt = f_i^d$ . Monotonicity of the forward rates is ensured by analyzing the sign of  $g$ . There are 3 regions where  $g$  need to be fixed to force monotonicity. We will focus our analysis on only one of them defined by  $g_{i-1} > 0, 0 > g_i > -\frac{1}{2}g_{i-1}$  or  $g_{i-1} < 0, 0 < g_i < -\frac{1}{2}g_{i-1}$  (called region (iii) in the authors paper), as well as the main region where  $g$  is already monotonic  $g_{i-1} > 0, -\frac{1}{2}g_{i-1} \geq g_i \geq -2g_{i-1}$  or  $g_{i-1} < 0, -\frac{1}{2}g_{i-1} \leq g_i \leq -2g_{i-1}$  (called region (i) in the authors paper). In region (iii), monotonicity is ensured by making  $g$  become:

$$g(x) = \begin{cases} g_i + (g_{i-1} - g_i) \left(\frac{\eta-x}{\eta}\right)^2 & \text{for } 0 < x < \eta \\ g_i & \text{for } \eta \leq x < 1 \end{cases} \quad (8)$$

with  $\eta = 3\frac{g_i}{g_i-g_{i-1}}$  to preserve the integral value. We can rewrite equation (8) as:

$$g(x) = \begin{cases} g_i + (g_{i-1} - g_i) \left(1 - x\frac{g_i-g_{i-1}}{3g_i}\right)^2 & \text{for } 0 < x < \eta \\ g_i & \text{for } \eta \leq x < 1 \end{cases} \quad (9)$$

Finally, let's define  $G$  as  $G' = g$ .

$$G(x) = g_i x + g_i \left(1 - \frac{x(g_i - g_{i-1})}{3g_i}\right)^3 \quad \text{for } 0 < x < \eta \quad (10)$$

### 2.2 Analytical Analysis

The stability criterion 1.3 can be measured by the change in discount factor following a change of input quote. For a tenor  $t \in [t_{i-1}, t_i]$ , the change in discount factor corresponds roughly to the change in  $G$  following a change of input quote  $f_i^d$ . We want the discount factor to vary smoothly when  $f_i^d$  is perturbed. The change in discount factor corresponds to the first derivative of  $G$  on  $f_i^d$ . For stability, we thus want the second derivative  $\frac{\partial^2 G}{\partial f_i^d{}^2}$  to be bounded.

Let's assume to simplify the problem that  $g_i$  is perturbed directly, instead of  $f_i^d$ . In reality, both  $g_{i-1}$  and  $g_i$  will be perturbed as a result of perturbing  $f_i^d$ , but the conclusions would be the same.

We suppose that  $x < \eta$  and do a Taylor expansion of  $\frac{\partial^2 G}{\partial g_i^2}$  in  $g_i$  around 0.

$$\frac{\partial^2 G}{\partial g_i^2}(x, g_i) = \frac{2}{9g_i^4}g_{i-1}^3x^3 - \frac{2}{9g_i^3}g_{i-1}^2x^3 + o(g_i^6) \quad (11)$$

There are 2 cases:

- (a)  $g_{i-1} \gg g_i$  (i.e. not around 0). As equation (11) is valid only for  $x \leq \eta$ ,  $x$  is at worst proportional to  $g_i$ . The first term will diverge as it is proportional to  $\frac{x^3}{g_i^4}$ . However it might not be that important in our case because  $G$  is evaluated from 0 to  $x$ , which is a tiny interval.
- (b)  $g_{i-1} \sim g_i$ . For example  $g_{i-1} = -4g_i$  initially.  $g_{i-1}$  is fixed, while we perturb  $g_i$ .  $\eta$  will stay around a fixed value: in this example  $\eta = \frac{3}{5}$ . The first term will diverge because it is of order  $\frac{g_{i-1}^3}{g_i^4}$ ,  $x$  is not small anymore, it can be around  $\eta$ .

Hagan and West computed a similar stability criterion ( $\frac{\partial f}{\partial f_i^d}$  is bounded) but only for the unconstrained spline in region (i), not globally. Even if they are careful to make  $g$  change smoothly at the boundary between region (i) and region (iii), we have shown that it is not enough and instability is possible when  $g_i$  and  $g_{i-1}$  are small.

### 2.3 Numerical Results

Does the analytical instability manifests itself in the real world? Unfortunately the answer is yes. We stumbled upon such a curve (Table 1). And it is that curve that led us to the analytical analysis. The annoying part is that this curve is nothing special, it does not have a forward approaching zero or a very strong convexity (see Figure 1). As a matter of fact, an unconstrained Hermite spline would fit all stability criteria on this curve.

Table 1.: Input Quotes and corresponding discount factors found by a global optimizer using raw interpolation.

(a)			(b)		
Quote Name	Type	Value	Term (Years)	Discount Factor	Discrete Forward
MM.USD.LIBOR.ON.T3750	Yield	0.205	0.008	0.999983	0.208%
MM.USD.LIBOR.TON.T3750	Yield	0.25	0.011	0.999977	0.208%
Future.USD.CME.EURODOLLAR.JUN.10	Future	99.6	0.378	0.998490	0.405%
Future.USD.CME.EURODOLLAR.SEP.10	Future	99.45	0.625	0.997115	0.559%
Future.USD.CME.EURODOLLAR.DEC.10	Future	99.205	0.871	0.995137	0.805%
Future.USD.CME.EURODOLLAR.MAR.11	Future	98.88	1.126	0.992266	1.134%
Future.USD.CME.EURODOLLAR.JUN.11	Future	98.52	1.375	0.988558	1.502%
Future.USD.CME.EURODOLLAR.SEP.11	Future	98.16	1.641	0.983680	1.861%
Swap.2Y.USD.LIBOR.6M/6M.T3750	Yield	1.2065	2.022	0.976031	2.050%
Swap.3Y.USD.LIBOR.6M/6M.T3750	Yield	1.796	3.019	0.947266	3.000%
Swap.5Y.USD.LIBOR.6M/6M.T3750	Yield	2.705	5.014	0.872068	4.147%
Swap.7Y.USD.LIBOR.6M/6M.T3750	Yield	3.28	7.025	0.790714	4.870%
Swap.10Y.USD.LIBOR.6M/6M.T3750	Yield	3.769	10.019	0.678050	5.133%

We look at the sequential delta and parallel delta for a simple money market trade of maturity 4.5 years (between our 3Y swap and 5Y swap). The sequential delta is computed by shifting each one of the curve underlying quotes by 1 basis point independently and calculating the price difference for our money market trade. The parallel delta is computed by shifting the whole curve by 1 basis point and calculating the price difference for our money market trade.

Table 2 shows that the sequential delta sum matches closely the parallel delta for all interpolation schemes except Hagan-West Monotone-Convex. The Monotone-Convex delta difference is around 12 times larger than expected. The main contributor to the error is the 5Y swap

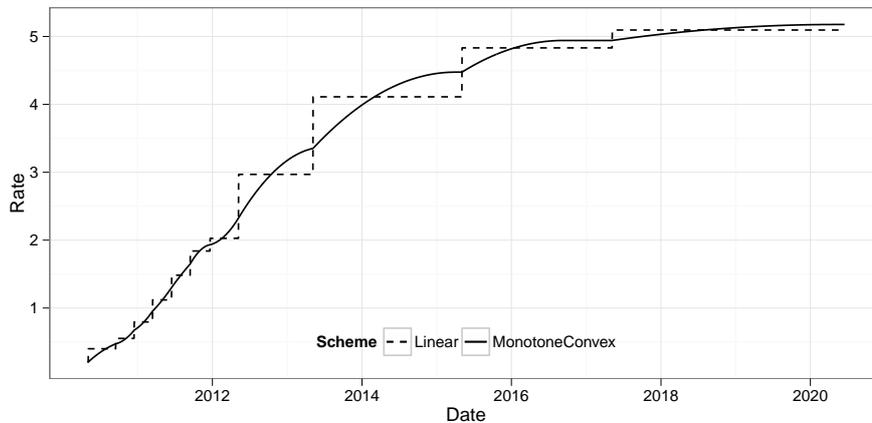


Figure 1.: 1-day forward curve

Table 2.: Sequential and Parallel delta for various curve interpolations corresponding to the curve in Figure 1.

Tenor	Monotone-Convex	Linear	Spline	Hyman89-Bessel	Harmonic
1D	\$(0.21)	\$(0.56)	\$(1.30)	\$(0.21)	\$(0.24)
M10	1.59	1.66	\$(3.11)	1.6	1.61
U10	1	0.99	\$(2.42)	1	1.03
Z10	0.9	0.79	\$(2.87)	0.9	0.92
H11	0.77	0.66	\$(3.06)	0.77	0.76
M11	0.59	0.48	\$(2.88)	0.6	0.62
U11	0.36	0.28	\$(5.14)	0.37	0.36
2Y	9.96	2.59	62.71	13.51	14.56
3Y	\$(66.59)	\$(67.30)	\$(109.81)	\$(72.55)	\$(78.04)
5Y	\$(480.54)	\$(391.27)	\$(446.42)	\$(451.32)	\$(439.21)
7Y	57.85	0	72.21	52.71	44.54
10Y	0.4	0	\$(10.82)	0.16	0.15
Sequential Delta Sum	\$(473.92)	\$(451.68)	\$(452.91)	\$(452.46)	\$(452.94)
Parallel Delta	\$(453.93)	\$(453.14)	\$(454.02)	\$(453.94)	\$(453.94)
Delta Difference	\$(19.99)	\$1.46	\$1.11	\$1.48	\$1.00

shift. When we interpolate the Monotone-Convex curve to find the discount factor at 4.5 year and analyze the Hagan-West algorithm in details, it turns out that we are in region (iii) with:

$$g_{i-1} = -0.007687, g_i = 0.003594, \eta = 0.95586$$

After a 1 basis point bump of the 5Y quote, we are still in region (iii) with

$$g_{i-1} = -0.007861, g_i = 0.003325, \eta = 0.89169$$

The cubic spline interpolation sequential delta shows some spill over especially for the 10Y swap as expected from a global method.

#### 2.4 Problem Reduction

We have been analyzing the interpolation role in the full curve construction mechanism. But the instability is actually only related to the interpolation of discount factors or forward rates. Let's analyse numerically only the interpolation part, independently of the curve construction.

Our input is now only the discount factors associated to each tenor, and we perturb the corresponding yield directly. At any given date, we want to know if the sum of the sequentially perturbed discount factors at this date is close to the discount factor obtained by shifting the yields in parallel. This corresponds indeed to our previous simple money market delta measure. If we had perturbed the discount factor directly, a parallel perturbation would create a bias in the short tenors.

We also want to look at the second derivative of the yield over the input yields  $\frac{\partial^2 y}{\partial y_i^2}(t)$ . This

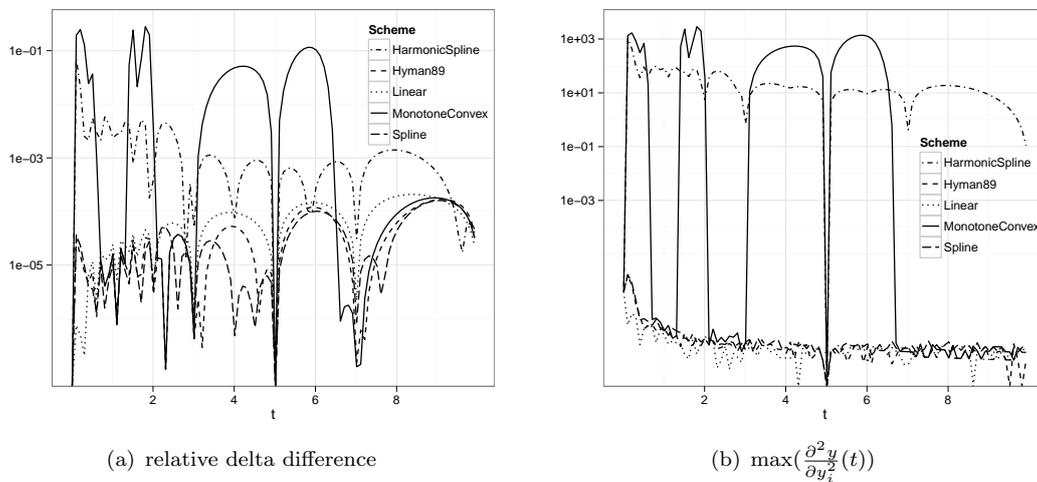


Figure 2.: 1 basis point yield perturbation for the curve defined in Table 1

is a similar measure as the  $\frac{\partial^2 G}{\partial g_i^2}(t)$  from equation 11.

Figure 2 shows that indeed, delta discrepancy and the second derivative of the yield are closely related. The Monotone Convex scheme delta discrepancy can be much worse than our previous example, up to 20% for a maturity between 1 and 2 years. Using a 5.9y maturity instead of a 4.5y maturity for the analysis of section 2.3 would have resulted in a higher delta discrepancy as well.

The log linear interpolation is the most stable one as expected, followed closely by the log spline and the log Hyman89 Bessel spline.

The cumulative Par-Point approach would provide by construction a consistent parallel delta. But it suffers from the same instability problems: the bucket delta varies a lot on a small curve bump (whether it is cumulative or a single quote) when  $\max(\frac{\partial^2 y}{\partial y_i^2}(t))$  is high, and the hedge is not stable. In addition, a global interpolation like a standard cubic spline presents similar spill-over.

### 3. Monotone Hermite Splines

#### 3.1 Piecewise Cubic Interpolation

Given the data  $z(t_0), z(t_1), \dots, z(t_n)$  with  $t_0 < t_1 < \dots < t_n$ , a piecewise cubic interpolant  $p$  is of the following form (De Boor, 2001), for  $i \in 0, \dots, n-1$ , for  $t \in [t_i, t_{i+1}]$ ,

$$p(t) = p_i(t) = c_0^i + c_1^i(t - t_i) + c_2^i(t - t_i)^2 + c_3^i(t - t_i)^3 \quad (12)$$

The interpolation conditions are:

$$p_i(t_i) = z(t_i), \quad p_i'(t_i) = s_i, \quad i = 0, \dots, n \quad (13)$$

$$p_i(t_{i+1}) = z(t_{i+1}), \quad p_i'(t_{i+1}) = s_{i+1}, \quad i = 0, \dots, n-1 \quad (14)$$

where  $s_i$  are free parameters. Let

$$D_i = \frac{z(t_{i+1}) - z(t_i)}{t_{i+1} - t_i} \quad (15)$$

The interpolation conditions give:

$$c_0^i(t_i) = z(t_i) \quad (16)$$

$$c_1^i = s_i \quad (17)$$

$$c_2^i = \frac{3D_i - s_{i+1} - 2s_i}{t_{i+1} - t_i} \quad (18)$$

$$c_3^i = -\frac{2D_i - s_{i+1} - s_i}{(t_{i+1} - t_i)^2} \quad (19)$$

For local interpolation schemes, the  $s_i$  are chosen so that the  $i$ th cubic polynomial depends only on information from, or near the interval  $[t_i, t_{i+1}]$ . While the  $s_i$  directly correspond to the slope of the interpolation at  $t_i$ , the  $D_i$  are just used as a convenient notation. We will later see that they actually have a financial interpretation.

Like Ametrano and Bianchetti (2009), we choose to interpolate on the logarithm of the discount factors, instead of the rates as Hagan and West (2006):  $z(t_i) = -y(t_i)t_i$ . This is particularly interesting when the interpolation preserves monotonicity: the forward rates will be guaranteed positive if the discount factors are decreasing.

### 3.2 Bessel spline with monotonic filter

In a Bessel spline, the  $s_i$  are chosen as the slope of the parabola interpolating the data at three consecutive data-points. Its order of accuracy is  $O(\delta t^2)$ .

$$s_i = \frac{(t_i - t_{i-1})D_i + (t_{i+1} - t_i)D_{i-1}}{t_{i+1} - t_{i-1}} \quad (20)$$

The resulting interpolant is then of class  $C^1$ , like Hagan-West interpolation. In practice, the forward will have "kinks" where the derivative is not continuous.

#### Hyman83 filter

Monotonicity can be ensured by applying some additional constraints on the  $s_i$ . Hyman shows that monotonicity is preserved if the derivative is bounded in the DeBoor-Schwartz region (Hyman, 1983):

$$s_i \leftarrow \begin{cases} \min[\max(0, s_i), 3 \min(|D_{i-1}|, |D_i|)] & \text{if } s_i \geq 0 \\ \max[\min(0, s_i), -3 \min(|D_{i-1}|, |D_i|)] & \text{if } s_i < 0, D_{i-1}D_i < 0 \end{cases} \quad (21)$$

#### Hyman89 filter

Those conditions are relaxed in (Dougherty *et al.*, 1989): let

$$m_i^{-1} = \frac{D_{i-1}(2(t_i - t_{i-1}) + t_{i-1} - t_{i-2}) - D_{i-2}(t_i - t_{i-1})}{t_i - t_{i-2}} \quad (22)$$

$$m_i^0 = \frac{D_{i-1}(t_{i+1} - t_i) + D_i(t_i - t_{i-1})}{t_{i+1} - t_{i-1}} \quad (23)$$

$$m_i^1 = \frac{D_i(2(t_{i+1} - t_i) + t_{i+2} - t_{i+1}) - D_{i+1}(t_{i+1} - t_i)}{t_{i+2} - t_i} \quad (24)$$

$$M_i = 3 \min(|D_{i-1}|, |D_i|, |m_i^0|, |m_i^1|) \quad (25)$$

If  $i > 2$  and  $m_i^0, m_i^{-1}, (D_{i-1} - D_{i-2}), (D_i - D_{i-1})$  have the same sign, let  $M_i \leftarrow \max(M_i, 1.5 \min(|m_i^0|, |m_i^{-1}|))$ . If  $i < n - 1$  and  $-m_i^0, -m_i^1, (D_i - D_{i-1}), (D_{i+1} - D_i)$  have

the same sign, let  $M_i \leftarrow \max(M_i, 1.5 \min(|m_i^0|, |m_i^1|))$ . Then we replace  $s_i$  with the following:

$$s_i \leftarrow \begin{cases} \operatorname{sgn}(s_i) \min(|s_i|, M_i) & \text{if } \operatorname{sgn}(s_i) = \operatorname{sgn}(m_i^0) \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

Applied on the log of the discount factors, the interpolation will preserve positivity of the forwards. Hagan and West had found that the forward could be negative with Hyman83 filter, because they applied the interpolation directly to the rates instead of the log of the discount factors. On their problematic curve, Ametrano and Bianchetti (2009) show that the Hyman83 monotonic interpolant on the log of the discount factors produces positive forwards as expected.

### 3.3 When monotonic filters break

All the previous monotonic filter share a common property: the first derivative is limited by a min or max function in order to ensure monotonicity. Unfortunately, as min and max don't have a continuous derivative at the region switch, this means that in turns, our discount factors have an exploding derivative at the switch, and our scheme is a priori not stable.

In order to illustrate this behavior, we modify slightly the previous curve so that it switches region when the input is perturbed.

Table 3.: Input Quotes and corresponding discount factors found by the global optimizer using raw interpolation.

(a)			(b)		
Quote Name	Type	Value	Term (Years)	Discount Factor	Discrete Forward
MM.USD.LIBOR.ON.T3750	Yield	0.205	0.008	0.999983	0.208%
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Swap.7Y.USD.LIBOR.6M/6M.T3750	Yield	3.28	7.025	0.787553	7.676%
Swap.10Y.USD.LIBOR.6M/6M.T3750	Yield	3.769	10.019	0.674985	5.151%

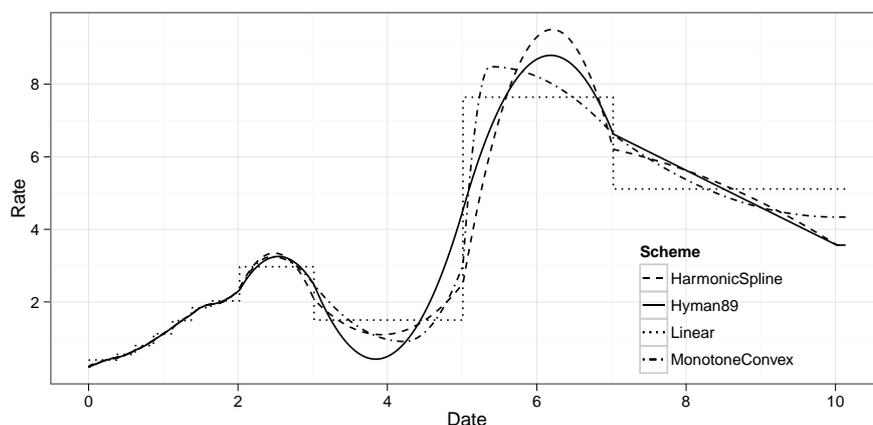


Figure 3.: 1-day forward curve

Table 3 and Figure 3 illustrate the curve for Hyman89 monotonic filter. The only change with the curve defined in Table 1 is the 5Y swap quote: we lowered it until the monotonic filter became active. Like in section 2.4, we focus at the interpolation stability, perturbing directly the yields.

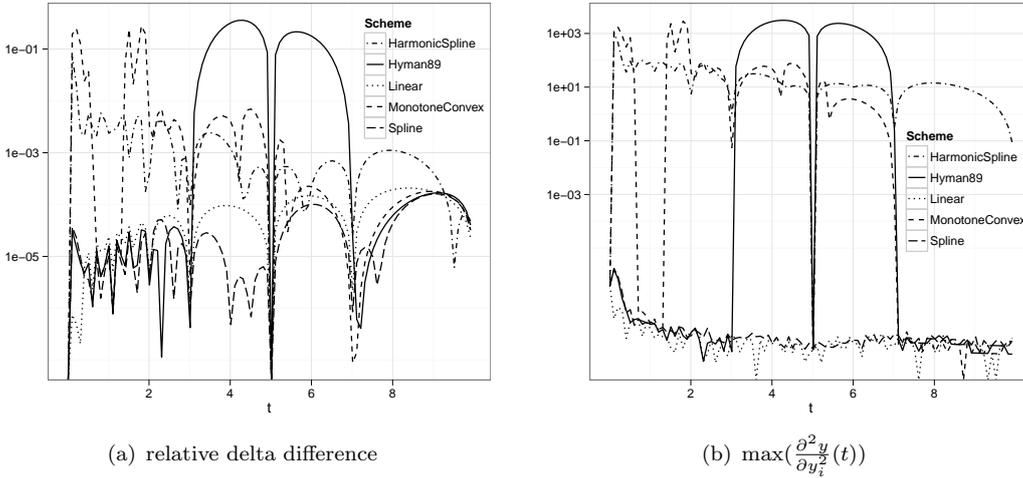


Figure 4.: 1 basis point yield perturbation for the curve defined in Table 3

Figure 4 shows that the sequential delta is very different from the parallel delta for the Bessel spline with Hyman89 filter, the difference being more than 20 times than the one of the raw interpolation. The Hyman83 filter will produce the same results on this curve as Hyman89. In this case, Hagan-West Monotone Convex interpolation has consistent delta. Again, delta discrepancy and the second derivative of the yield are closely related.

There are various alternatives in the same spirit as Hyman filtering: Steffen proposes a stricter filter for the case of the Bessel spline (Steffen, 1990), and many other limiters are reviewed in (Huynh, 1993). Similarly to the Hyman case, we have found curves producing a large delta discrepancy for each of those involving a min or max function.

### 3.4 Harmonic spline to the rescue

Brodie proposed to define  $s_i$  in equation (13) by a weighted harmonic mean (Fritsch and Butland, 1984)<sup>1</sup>:

$$\frac{1}{s_i} = \frac{t_i - t_{i-1} + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})} \frac{1}{D_{i-1}} + \frac{2(t_i - t_{i-1}) + t_{i+1} - t_i}{3(t_{i+1} - t_{i-1})} \frac{1}{D_i} \text{ if } D_{i-1}D_i > 0 \quad (27)$$

This is mathematically equivalent to Fritsch PCHIM formulation part of Slatec Fortran library:

$$s_i = \frac{\min(|D_{i-1}|, |D_i|)}{\frac{2(t_i - t_{i-1}) + t_{i+1} - t_i}{3(t_{i+1} - t_{i-1})} \frac{D_{i-1}}{\max(|D_{i-1}|, |D_i|)} + \frac{t_i - t_{i-1} + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})} \frac{D_i}{\max(|D_{i-1}|, |D_i|)}} \text{ if } D_{i-1}D_i > 0 \quad (28)$$

and  $s_i = 0$  if  $D_{i-1}D_i \leq 0$ .

<sup>1</sup>not to be confused with the sometimes called Fritsch-Butland algorithm presented in (Hyman, 1983) and available in Quantlib

The boundaries are defined by:

$$s_0 = \frac{2(t_1 - t_0) + (t_2 - t_1)}{t_2 - t_0} D_0 - \frac{t_1 - t_0}{t_2 - t_0} D_1 \quad (29)$$

$$s_n = -\frac{t_n - t_{n-1}}{t_n - t_{n-2}} D_{n-2} + \frac{2(t_n - t_{n-1}) + t_{n-1} - t_{n-2}}{t_n - t_{n-2}} D_{n-1} \quad (30)$$

and then filtered for monotonicity according to

$$s_0 \leftarrow \begin{cases} 3D_0 & \text{if } s_0 D_0 > 0, D_0 D_1 \leq 0, |s_0| < |3D_0| \\ s_0 & \text{else if } s_0 D_0 > 0 \\ 0 & \text{if } s_0 D_0 \leq 0 \end{cases} \quad (31)$$

$$s_n \leftarrow \begin{cases} 3D_{n-1} & \text{if } s_n D_{n-1} > 0, D_{n-1} D_{n-2} \leq 0, |s_n| < |3D_{n-1}| \\ s_n & \text{else if } s_n D_{n-1} > 0 \\ 0 & \text{if } s_n D_{n-1} \leq 0 \end{cases} \quad (32)$$

Except at the boundaries, this interpolation has continuous derivatives with regards to the  $z_i$  and will thus be stable for the use of yield curve interpolation on the log of the discount factors, it is of class  $C^1$ . There will be no big spike in  $\frac{\partial^2 y}{\partial y_i^2}$  unlike Hyman filtering in Figure 4. This is at the cost of a generally noisier second derivative, and as a consequence, the mean relative delta error is higher on data where Hyman works well, like on Figure 2, but it remains under 1%.

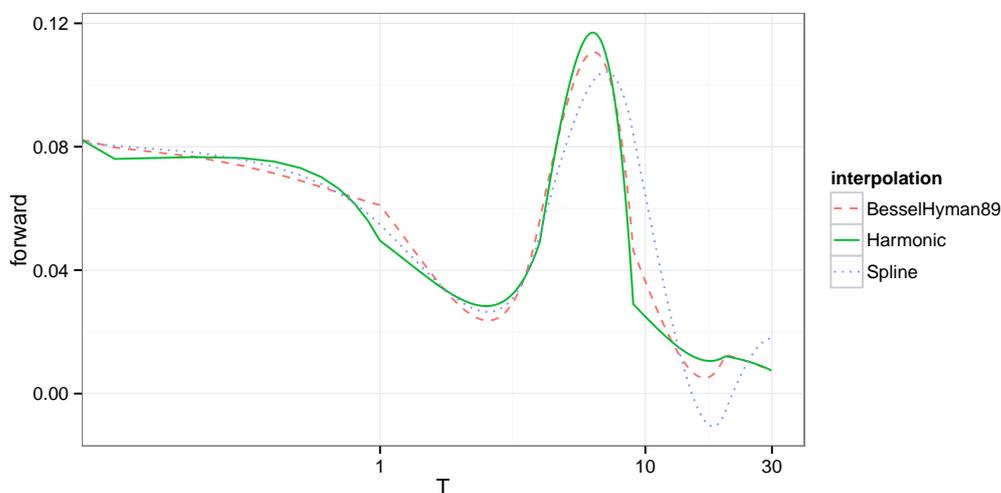


Figure 5.: 1-day forward curve using Harmonic Spline interpolation

We take the example of a curve where all cubic interpolation methods studied in (Hagan and West, 2006) produce negative forward rates, also studied in (Ametrano and Bianchetti, 2009). As seen on Figure 5 the Harmonic spline interpolation on the logarithm of the discount factors does not produce negative forwards.

When  $t_i - t_{i-1} = t_{i+1} - t_i$ , letting  $r = \frac{D_i}{D_{i-1}}$ , we have the following formula:  $s_i = D_i \frac{r+|r|}{1+|r|}$ . This is the Van Leer limiter (Van Leer, 1974). Among the many limiters described in (Huynh, 1993), beside the Van Leer limiter, we have found one that does not rely on min or max functions and behaves well on our tests curves. It relies on a rational function tailored to

ensure continuity in the first derivative when  $D_i = D_{i-1}$  and can be expressed as:

$$s_i = \frac{3D_i D_{i-1} (D_i + D_{i-1})}{D_i^2 + 4D_i D_{i-1} + D_{i-1}^2} \quad (33)$$

when  $D_i D_{i-1} > 0$  and  $s_i = 0$  otherwise.

Like the harmonic mean, it produces a  $C^1$  stable interpolation and behaves very similarly on our test curves.

### 3.5 Harmonic Forwards

There is an equivalence between Hagan-West approach of interpolating on discrete forwards and an interpolation on the log of the discount factors: the Hagan-West interpolation without any monotonicity and convexity constraints is a Bessel spline interpolation with specific boundaries. Using  $z(t_i) = -y(t_i)t_i$  in equation (15) and by the definition of a discrete forward rate, we have:

$$f_{i+1}^d = -D_i \quad (34)$$

Equations (5) and (20) give

$$f_i = -s_i \quad (35)$$

The harmonic spline idea can be applied directly to the forwards by replacing the average forward definition in Equation (5) with the following:

$$\frac{1}{f_i} = \frac{t_i - t_{i-1} + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})} \frac{1}{f_i^d} + \frac{2(t_i - t_{i-1}) + t_{i+1} - t_i}{3(t_{i+1} - t_{i-1})} \frac{1}{f_{i+1}^d} \text{ if } f_i^d f_{i+1}^d > 0 \quad (36)$$

One can then just follow on and use Hagan-West algorithm without explicitly enforcing any specific monotonicity and convexity constraints as, by construction, it will be already monotonic. One advantage of following this approach is speed, another is a better sense of the boundary conditions meaning as they directly relate to forward rates.

## 4. Tension Splines

Tension Splines are considered in (Andersen, 2005) to build a yield curve with control on locality and convexity. They are also market standard for some curves: to build a designated yield curve, SIFMA (Securities Industry and Financial Markets Association) defines the spread adjustments using a  $C^2$  tension spline on the yields of designated bills and designated notes (Securities Industry and Financial Markets Association, 2004).

There are several formulations for tension splines. The most common is the  $C^2$  hyperbolic tension spline (Pruess, 1978) is defined as following. Given the data  $z(t_0), z(t_1), \dots, z(t_n)$  with  $t_0 < t_1 < \dots < t_n$ , and a set of tension parameters  $\rho_0, \rho_1, \dots, \rho_{n-1}$ ,  $p$  is a tension spline corresponding to this data and tension parameters if:

$$\begin{aligned} p^{(4)} - \rho_i^2 p'' &= 0 \text{ in each } [t_i, t_{i+1}] \text{ for } i \in 0, \dots, n-1 \\ p &\in C^2[t_0, t_n] \end{aligned}$$

The interpolation conditions are:

$$p(t) = p_i(t), \quad t \in [t_i, t_{i+1}] \quad (37)$$

$$p_i(t_i) = z(t_i), \quad i = 0, \dots, n \quad (38)$$

$$p_i(t_{i+1}) = z(t_{i+1}), \quad p_{i+1}''(t_i) = p_i''(t_i), \quad i = 0, \dots, n-1 \quad (39)$$

The interpolant  $p$  is of the following form: for  $i \in 0, \dots, n-1$ , for  $t \in [t_i, t_{i+1}]$ ,

$$p(t) = p_i(t) = p_i(t_i)\lambda_i(t_{i+1} - t) + p_{i+1}(t_{i+1})\lambda_i(t - t_i) + p_i''\sigma_i(t_{i+1} - t) + p_{i+1}''\sigma_i(t - t_i) \quad (40)$$

with

$$h_i = t_{i+1} - t_i \quad (41)$$

$$\lambda_i(t) = t/h_i \quad (42)$$

$$\sigma_i(t) = \frac{1}{\rho_i^2} \left( \frac{\sinh(\rho_i t)}{\sinh(\rho_i h_i)} - \lambda_i(t) \right) \quad (43)$$

$$p_i'' = p''(t_i) \quad (44)$$

and  $p_i''$  defined by the following tridiagonal system (continuity at  $p''$ ):

$$c_{i-1}p_{i-1}'' + (b_{i-1} + b_i)p_i'' + c_i p_{i+1}'' = \frac{p_{i+1}(t_{i+1}) - p_i(t_i)}{h_i} - \frac{p_i(t_i) - p_{i-1}(t_{i-1})}{t_{i-1}} \quad (45)$$

where

$$c_i = \frac{1}{\rho_i^2} \left( \frac{1}{h_i} - \frac{\rho_i}{\sinh(\rho_i h_i)} \right), \quad (46)$$

$$b_i = \frac{1}{\rho_i^2} \left( \rho_i \frac{\cosh(\rho_i h_i)}{\sinh(\rho_i h_i)} - \frac{1}{h_i} \right) \quad (47)$$

$$(48)$$

$p_0''$  and  $p_n''$  can be user defined. Setting them to 0 leads to the so called natural boundary condition.

#### 4.1 $C^1$ Tension Splines

Renka extended the definition to  $C^1$  splines (Renka, 1987), allowing the user to give first derivatives estimates at the knots. This can produce a local interpolation. But unfortunately, the derivative estimates suffer from the same drawbacks as the hermite splines derivative estimates. Renka suggests the use of derivatives estimates that preserve monotonicity. Hyman filtering would produce the same sort of instabilities as in section 3.3. Furthermore the resulting tension spline is very close to an ordinary Hermite spline, even when convexity is preserved via Renka methodology. The  $C^1$  tension splines do not seem to offer any significant advantage.

#### 4.2 $C^2$ Tension

Let's first consider the case of a uniform tension, as used for spread adjustments curves. How do we select the tension parameter?

A natural choice for yield curve construction would be to start with a parameter that preserves monotonicity and convexity using an algorithm like in (Pruess, 1978). This is how we found our tension parameter for the curves in Figures 6 and 7. Figure 6 shows that a tension spline interpolant preserving monotonicity and convexity creates a curve not too far from a simple raw interpolation curve with our uniform tension definition.

If we simply scale the tension by  $t_{i+1} - t_i$ , the tension will become  $\frac{\rho_i}{t_{i+1} - t_i}$  and the interpolant will be independent of a scaling of the abscissae (Renka, 1987). This creates a much smoother curve (the "Scaled" curve in Figure 6). One has to be careful not to optimize the tension for convexity and monotonicity everytime the curve is built, otherwise the curve is very unstable

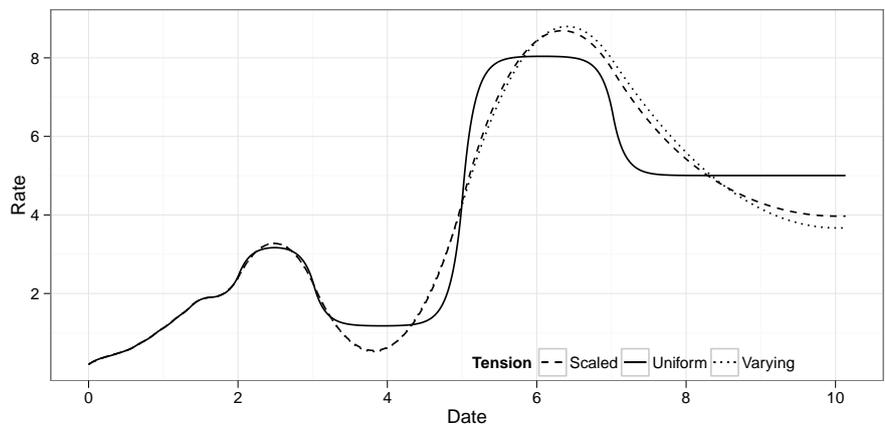
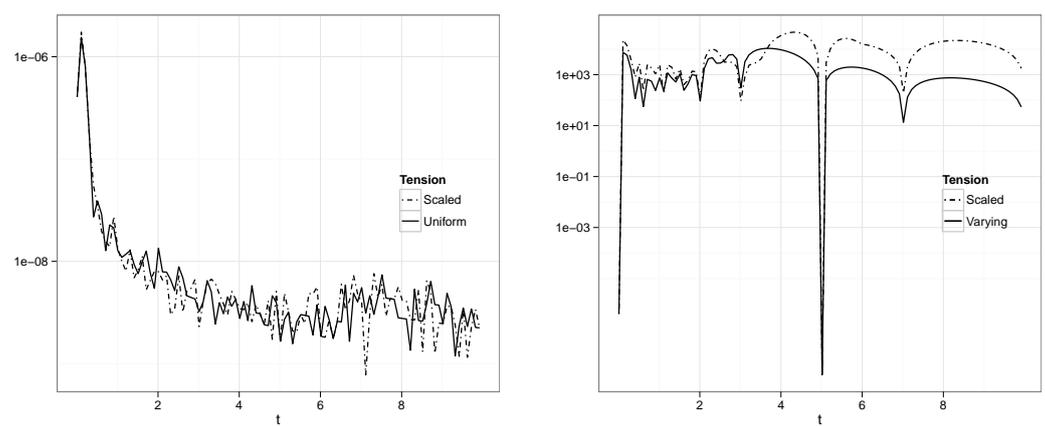


Figure 6.: 1-day forward curve using Tension Splines interpolation for the curve defined by Table 3



(a)  $\max(\frac{\partial^2 y}{\partial t^2}(t))$  keeping the same tension in perturbations (b)  $\max(\frac{\partial^2 y}{\partial t^2}(t))$  reoptimizing tensions in perturbations

Figure 7.: 1 basis point yield perturbation for the curve defined in Table 3

under perturbation (Figure 7). This is where the use of tension spline can be problematic: each time the tension parameter is updated, the risk measures can change a lot.

Many suggest the use of a different tension at each knot (Pruess, 1978; Renka, 1987; Andersen, 2005) (the “Varying” curve in Figure 6) . The stability issues will be more localised if a few neighbouring tensions are changed instead of all tensions. But this makes even more parameters to tweak. It is unclear how this can be done without an automatic procedure. And still the delta will be unreliable when a tension is updated around this tension tenors.

If one optimizes only for convexity and monotonicity, the tension spline can be very close to a regular cubic spline on some inputs, this is what happens on the curve defined in Table 1. The curve will be globally sensitive to perturbation and present just as much leaking as a cubic spline. One should consider adding specific locality constraints. One possibility is the use of cubic tension B-splines (Andersen, 2005).

## 14 REFERENCES

## 5. Conclusion

We have shown that Hagan and West monotone convex interpolation does not always produce a stable yield curve, especially under sequential shifts. Tension splines produce nice looking yield curves. But those curves are very unstable when the tension parameters are updated. Among Hermite splines, any monotonic limiter (e.g. Hyman) will produce unstable yield curves. Only the Harmonic spline produces a stable local interpolation while preserving monotonicity and convexity of the discount factors. Furthermore as the Hagan and West interpolation on discrete forward rates is equivalent to some Hermite spline interpolation on the logarithm of the discount factors, the Harmonic spline approach can also be directly to the forward rates, resulting in a potentially faster and more intuitive algorithm.

## References

- Ametrano, F. and Bianchetti, M. (2009) Bootstrapping the Illiquidity, *Modelling Interest Rates: Advances for Derivatives Pricing*.
- Andersen, L. B. (2005) Yield Curve Construction with Tension Splines, *SSRN eLibrary*.
- Andersen, L. and Piterbarg, V. (2010) *Interest Rate Modeling - Volume I: Foundations and Vanilla Models*, (Atlantic Financial Press).
- De Boor, C. (2001) *A practical guide to splines*, (Springer Verlag).
- Dougherty, R., Edelman, A. and Hyman, J. (1989) Nonnegativity-, monotonicity-, or convexity-preserving cubic and quintic Hermite interpolation, *Mathematics of Computation*, 52(186), pp. 471–494.
- Fritsch, F. and Butland, J. (1984) A method for constructing local monotone piecewise cubic interpolants, *SIAM journal on scientific and statistical computing*, 5, p. 300.
- Hagan, P. and West, G. (2006) Interpolation methods for curve construction, *Applied Mathematical Finance*, 13(2), pp. 89–129.
- Huynh, H. T. (1993) Accurate monotone cubic interpolation, *SIAM Journal on Numerical Analysis*, 30(1), pp. 57–100.
- Hyman, J. (1983) Accurate monotonicity preserving cubic interpolation, *SIAM J. Sci. Stat. Comput.*, 4(4), pp. 645–654.
- Pruess, S. (1978) An algorithm for computing smoothing splines in tension, *Computing*, 19(4), pp. 365–373.
- Renka, R. (1987) Interpolatory tension splines with automatic selection of tension factors, *SIAM Journal on Scientific and Statistical Computing*, 8, p. 393.
- Securities Industry and Financial Markets Association, 2004 Practice Guidelines for Trading in GSE European Callable Securities - Appendix C - The BMA Designated Yield Curve. (2004) , Technical report.
- Steffen, M. (1990) A simple method for monotonic interpolation in one dimension, *Astronomy and Astrophysics*, 239, p. 443.
- Van Leer, B. (1974) Towards the ultimate conservative difference scheme. II. Monotonicity and conservation combined in a second-order scheme, *Journal of computational physics*, 14(4), pp. 361–370.
- Wolberg, G. and Alf, I. (1999) Monotonic cubic spline interpolation. In: *Computer Graphics International, 1999. Proceedings*, pp. 188–195.

## Appendix A. Harmonic Spline in Scilab

Scilab, an open source software for numerical computation includes the PCHIM algorithm. Here is an example of how to interpolate a given zero rates term structure with Scilab:

Listing 1: Scilab code for harmonic spline interpolation

```
terms = [0.011111 0.105556 0.269444 0.525000 0.775000 ..
         1.027778 1.544444 2.050000 3.063889 4.077778 ..
         5.086111 7.125000 10.163889];
zerorates = [0.0305947992 0.0308287918 0.0310541511 0.031233188 ..
             0.0323659069 0.0341521253 0.0394031712 0.0422235447 ..
             0.0493163764 0.0543568346 0.0575012361 0.0613722734 ..
             0.0653758336];
logdf = -zerorates.*terms;
d = splin(terms, logdf, "monotone");
t = linspace(0, 10, 101);
```

```
[z z1] = interp(t, terms, logdf, d, "natural");  
forward = -z1; //instantaneous forward  
plot2d(t, forward)
```